An optimal stopping mean-field game of resource sharing

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Outline

How do economic agents adapt to climate change?

- Water security is one of the most tangible and fastest-growing social, political and economic challenges faced today
- The coal industry is an important consumer of freshwater resources and is responsible for 7% of all water withdrawal globally
- Cooling power plants are responsible for the greatest demand in fresh water

A model for producers competing for a scarce resource

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- Each producer faces demand level M_t^i and can produce up to M_t^i if the water supply allows:
  - With technology 1, one unit of water is required to produce one unit of good;
  - With technology 2, no water is required.

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  - With technology 2, no water is required.
- In case of shortage of water, the available supply is shared among producers according to their demand levels.

A model for producers competing for a scarce resource

- The demand of i-th producer follows the dynamics

$$\frac{dM_t^i}{M_t^i} = \mu dt + \sigma dW_t^i, \quad M_0^i = m^i.$$

where $W^1, \dots, W^N$ are independent Brownian motions.

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- With technology 2, the output is M_t^i and with technology 1 the output is

$$Q_t^i = \begin{cases} M_t^i, & \text{if } \widetilde{Z}_t \geq \sum_{i=1}^N M_t^j \mathbf{1}_{\tau_j > t} \\ \frac{\widetilde{Z}_t}{\sum_{j=1}^N M_t^j \mathbf{1}_{\tau_j > t}} M_t^i & \text{otherwise.} \end{cases}$$

$\Rightarrow Q_t^i = \omega_t^N M_t^i$, where ω_t^N is the proportion of demand which may be satisfied

$$\omega_t^N = \frac{\widetilde{Z}_t}{\sum_{j=1}^N M_t^j \mathbf{1}_{\tau_j > t}} \wedge 1.$$

Cost function of producers

The cost function of the producer is given by

$$\int_0^{\tau_i} e^{-\rho t} p Q_t^i dt - \int_0^{\tau_i} e^{-\rho t} \hat{p}(M_t^i - Q_t^i) dt - e^{-\rho \tau_i} K + \int_{\tau_i}^{\infty} e^{-\rho t} \tilde{p} M_t^i dt$$
$$= \int_0^{\tau_i} e^{-\rho t} p \omega_t^N M_t^i dt - \int_0^{\tau_i} e^{-\rho t} \hat{p}(1 - \omega_t^N) M_t^i dt - e^{-\rho \tau_i} K + \int_{\tau_i}^{\infty} e^{-\rho t} \tilde{p} M_t^i dt$$

where we assume that $\rho > \mu$.

p is the gain from producing with technology 1;

$\hat{p}$ is the penalty paid for not meeting the demand;

K is the cost of switching the technology;

$\tilde{p}$ is the gain from producing with technology 2.

Outline

Mean-field games

Introduced by Lasry and Lions (2006,2007) and Huang, Caines and Malhamé (2006) to describe large-population games with symmetric interactions.

- Each player controls its state $X_t^i \in \mathbb{R}^d$ by taking an action $\alpha_t^i \in A \subset \mathbb{R}^k$:

$$dX_t^i = b(t, X_t^i, \bar{\mu}^{N-1}_{X_t^{-i}}, \alpha_t^i)dt + \sigma(t, X_t^i, \bar{\mu}^{N-1}_{X_t^{-i}}, \alpha_t^i)dW_t^i,$$

W^i are independent and $\bar{\mu}^{N-1}_{X_t^{-i}}$ is the empirical distribution of other players.

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  W^i are independent and $\bar{\mu}^{N-1}_{X_t^{-i}}$ is the empirical distribution of other players.

- Each player minimises the cost

$$J^i(\boldsymbol{\alpha}) = \mathbb{E}\left[\int_0^T f(t, X_t^i, \bar{\mu}^{N-1}_{X_t^{-i}}, \alpha_t^i)dt + g(X_T^i, \bar{\mu}^{N-1}_{X_T^{-i}})\right],$$

- We look for a Nash equilibrium: $\hat{\boldsymbol{\alpha}} \in A^N$: $\forall i$, $\forall \alpha^i \in A$, $J^i(\hat{\boldsymbol{\alpha}}) \le J^i(\alpha^i, \hat{\boldsymbol{\alpha}}^{-i})$.

Mean-field games

As $N \to \infty$, it is natural to assume that $\bar{\mu}^{N-1}_{X_t^{-i}}$ converges to a deterministic distribution; Nash equilibrium is described as follows (Carmona and Delarue '17):

- The representative player controls its state X^α depending on the deterministic flow $(\mu_t)_{0\leq t\leq T}$:

$$dX_t^\alpha = b(t, X_t^\alpha, \mu_t, \alpha_t)dt + \sigma(t, X_t^\alpha, \mu_t, \alpha_t)dW_t.$$

- It minimises the cost

$$\inf_{\alpha\in A} J^\mu(\alpha), \quad J^\mu(\alpha) = \mathbb{E}\left[\int_0^T f(t, X_t^\alpha, \mu_t, \alpha_t)dt + g(X_T^\alpha, \mu_T)\right] \qquad (*)$$

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- We look for a flow $(\mu_t)_{0\leq t\leq T}$ such that $\mathcal{L}(\hat{X}_t^\mu) = \mu_t$, $t \in [0, T]$, where $\hat{X}^\mu$ is the solution to $(*)$.

The analytic approach

The stochastic control problem is characterized as the solution to a HJB equation

$$\partial_t V + \max_\alpha \left\{ f(t,x,\mu_t,\alpha) + b(t,x,\mu_t,\alpha)\partial_x V + \frac{1}{2}\sigma^2(t,x,\mu_t,\alpha)\partial^2_{xx} V \right\} = 0$$

with the terminal condition $V(T,x) = g(x,\mu_T)$.

The flow of densities solves the Fokker-Planck equation

$$\partial_t \mu_t - \frac{1}{2}\partial^2_{xx}(\sigma^2(t,x,\mu_t,\hat{\alpha}_t)\mu_t) + \partial_x(b(t,x,\mu_t,\hat{\alpha}_t)\mu_t) = 0,$$

with the initial condition $\mu_0 = \delta_{X_0}$, where $\hat{\alpha}$ is the optimal feedback control.

$\Rightarrow$ A coupled system of a Hamilton-Jacobi-Bellman PDE (backward) and a Fokker-Planck PDE (forward)

Outline

Optimal stopping mean-field games

In optimal stopping mean-field games (aka MFG of timing), the strategy of each agent is a stopping time.

- Nutz (2017): bank run model with common noise, interaction through proportion of stopped players, explicit form of optimal stopping time;
- Carmona, Delarue and Lacker (2017): a general timing game with common noise, interaction through proportion of stopped players. Existence of strict equilibria under complementarity condition (others leaving create incentive for me to leave), no uniqueness.
- Bertucci (2017): Markovian state of each agent; no common noise, interaction through density of states of players still in the game, analytic approach (obstacle problem), existence of mixed equilibria, uniqueness under antimonotonicity condition (others leaving create incentive for me to stay).

The model

Consider n agents X^i, $i = 1, \dots, n$ with dynamics

$$dX_t^i = \mu(t, X_t^i)dt + \sigma(t, X_t^i)dW_t^i,$$

where W^i, $i = 1, \dots, n$ are independent and μ and σ are Lipschitz with linear growth in X, uniformly on $t \in [0, T]$.

We denote by $\mathcal{L}$ the infinitesimal generator:

$$\mathcal{L}f(t,x) = \mu(t,x)\frac{\partial f}{\partial x}(t,x) + \frac{\sigma^2(t,x)}{2}\frac{\partial f^2}{\partial x^2}(t,x).$$

The single-agent problem

Each agent aims to solve the optimal stopping problem

$$\max_{\tau_i \in \mathcal{T}([0,T])} \mathbb{E}[\int_0^\tau e^{-\rho t} \tilde{f}(t, X_t^i, m_t^n) dt + e^{-\rho \tau} g(\tau, X_\tau^i)],$$

where

$$m_t^n(dx) = \frac{1}{n} \sum_{i=1}^n \delta_{X_t^i}(dx) \mathbf{1}_{t \leq \tau_t}.$$

Letting $f(t,x,\mu) = e^{-\rho t}(\tilde{f}(t,x,\mu) - \rho g(t,x) + \frac{\partial g}{\partial t} + \mathcal{L} g)$ the problem becomes

$$\max_{\tau_i \in \mathcal{T}([0,T])} \mathbb{E}[\int_0^\tau f(t, X_t^i, m_t^n) dt].$$

The MFG formulation: optimal stopping problem

As $N \to \infty$, we expect m_t^n to converge to a deterministic limit m_t, $\forall t \in [0, T]$.

The state of the representative agent with initial value x follows the dynamics

$$dX_t^x = \mu(t, X_t^x)dt + \sigma(t, X_t^x)dW_t.$$

and the optimal stopping problem for the agent takes the form

$$\max_{\tau \in \mathcal{T}([0,T])} \mathbb{E}[\int_0^\tau f(t, X_t^x, m_t)dt].$$

The MFG formulation: optimal stopping problem

Let $\tau^{m,x}$ be the optimal stopping time for agent with initial demand level x.

Given initial measure m_0^* we look for $(m_t)_{0\leq t\leq T}$ s.t.

$$m_t(A) = \int m_0^*(dx)\mathbb{P}[X_t^x \in A; \tau^{m,x} > t], \quad A \in \mathcal{B}(\mathbb{R}),\ t \in [0,T]. \qquad (1)$$

Solution of optimal stopping MFG: fixed point of the right-hand side of (1).

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Pure solutions (stopping-time based) do not always exist (Bertucci '2017) $\Rightarrow$ we consider relaxed solutions.

$\Rightarrow$ agents may stay in the game after the optial stopping time if this does not decrease their value.

Outline

Relaxed optimal stopping

Inspired by works on linear programming formulation of stochastic control, e.g., Stockbridge '90; El Karoui, Huu Nguyen and Jeanblanc '87 and more recently Bukhdahn, Goreac and Quincampoix '11. Application to MFG in Lacker '15.

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Consider the optimal stopping problem

$$\sup_{\tau \in \mathcal{T}([0,T])} \mathbb{E}\left[\int_0^\tau f(t, X_t)dt\right], \qquad X_t = x + \int_0^t \mu(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s$$

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Introduce occupation measure $m_t(A) := \mathbb{E}[\mathbf{1}_A(X_t)\mathbf{1}_{t\leq\tau}]$. The objective writes

$$\int_{[0,T]\times\Omega} f(t,x)m_t(dx)\, dt.$$

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By Itô formula, for positive, regular test function u,

$$u(0,x)+\int_{[0,T]\times\Omega}\left(\frac{\partial u}{\partial t}+\mu\frac{\partial u}{\partial x}+\frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial x^2}\right)m_t(dx)\,dt = \mathbb{E}[u(\tau\wedge T, X_{\tau\wedge T})]\geq 0.$$

Relaxed optimal stopping

For a given initial distribution m_0^*, compute

$$V^R(m_0^*) = \sup_{m \in \mathcal{A}(m_0^*)} \int_0^T \int_\Omega f(t,x) m_t(dx)\, dt.$$

where the set $\mathcal{A}(m_0^*)$ contains all families of positive bounded measures $(m_t)_{0 \leq t \leq T}$ on Ω, satisfying

$$\int_\Omega u(0,x) m_0^*(dx) + \int_0^T \int_\Omega \left\{ \frac{\partial u}{\partial t} + \mathcal{L}u \right\} m_t(dx)\, dt \geq 0$$

for all $u \in C^{1,2}([0,T] \times \Omega)$ such that $u \geq 0$ and $\frac{\partial u}{\partial t} + \mathcal{L}u$ is bounded.

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for all $u \in C^{1,2}([0,T] \times \Omega)$ such that $u \geq 0$ and $\frac{\partial u}{\partial t} + \mathcal{L}u$ is bounded.

$\Rightarrow$ In other words, $-\frac{\partial m}{\partial t} + \mathcal{L}^* m \geq 0$ in the sense of distributions.

Link to the strong formulation

- Under standard assumptions (including ellipticity, see Bensoussan-Lions '82), $V^R(\delta_x) = v(0,x)$, where

$$v(t,x) = \sup_{\tau \in \mathcal{T}([t,T])} \mathbb{E}\left[\int_t^\tau f(s, X_s^{(t,x)})ds\right].$$

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- Let $\hat{m}$ be any solution of the relaxed optimal stopping problem. Then,

$$\int_{(t,x)\in[0,T]\times\Omega : v(t,x)=0} |f(t,x)|\hat{m}_t(dx) = 0$$

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- For test functions u such that $\operatorname{supp} u \in \{(t,x) \in [0,T] \times \Omega : v(t,x) > 0\}$,

$$\int_\Omega u(0,x)m_0^*(dx) + \int_0^T \int_\Omega \left\{\frac{\partial u}{\partial t} + \mathcal{L}u\right\} \hat{m}_t(dx)\, dt = 0.$$

$\Rightarrow$ $\hat{m}$ satisfies Fokker-Planck on $\{v > 0\}$.

Relaxed optimal stopping: existence

Let V be the space of families of positive measures on Ω $(m_t(dx))_{0\leq t\leq T}$ such that $\int_0^T \int_\Omega m_t(dx)\, dt < \infty$.

To each $m \in V$, associate a positive measure on $[0, T] \times \Omega$ defined by $\mu(dt, dx) := m_t(dx)\, dt$, and endow V with the topology of weak convergence.

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Lemma (Compactness)

Let m_0^* be a bounded positive measure satisfying

$$\int_\Omega \ln\{1+|x|\} m_0^*(dx) < \infty.$$

Then the set $\mathcal{A}(m_0^*)$ is weakly compact.

Relaxed optimal stopping: existence

Lemma (Existence for relaxed optimal stopping)

Let m_0^* satisfy the compactness condition and assume that f is of the form

$$f(t,x) = \sum_{i=1}^{n} \bar{f}_i(t) g_i(x)$$

where g_i is a difference of two convex functions whose derivatives have polynomial growth and $\bar{f}_i$ is bounded measurable.

Then there exists $m^* \in \mathcal{A}(m_0^*)$ which maximizes the functional

$$m \mapsto \int_0^T \int_\Omega f(t,x) m_t(dx)\, dt$$

over all $m \in \mathcal{A}(m_0^*)$.

Relaxed optimal stopping MFG

Definition (Nash equilibrium)

Given the initial distribution m_0^*, a family of measures $m^* \in \mathcal{A}(m_0^*)$ is a Nash equilibrium for the relaxed MFG optimal stopping problem if

$$\int_0^T \int_\Omega f(t,x,m_t^*) m_t(dx)\,dt \leq \int_0^T \int_\Omega f(t,x,m_t^*) m_t^*(dx)\,dt,$$

for all $m \in \mathcal{A}(m_0^*)$.

$\Rightarrow$ the set of Nash equilibria coincides with the set of fixed points of the set-valued mapping $G : \mathcal{A}(m_0^*) \to \mathcal{A}(m_0^*)$ defined by

$$G(m) = \operatorname{argmax}_{\hat{m}\in\mathcal{A}(m_0^*)} \int_0^T \int_\Omega f(t,x,m_t)\hat{m}_t(dx)\,dt,$$

Optimal stopping MFG: existence

Theorem

Let m_0^* satisfy the compactness condition, and let f be of the form

$$f(t,x,m) = \sum_{i=1}^{n} \bar{f}_i \left(t, \int_\Omega \bar{g}_i(x) m_t(dx) \right) g_i(x),$$

where g_i and $\bar{g}_i$ can be written a difference of two convex functions whose derivatives have polynomial growth, and $\bar{f}_i$ is bounded measurable and continuous with respect to its second argument.

Then there exists a Nash equilibrium for the relaxed MFG problem.

Proof: Fan-Glicksberg fixed point theorem for set-valued mappings.

Optimal stopping MFG: uniqueness

Let

$$f(t,x,m) = \bar{f}_1\left(t, \int_\Omega g_1(x) m_t(dx)\right) g_1(x) + \bar{f}_2(t) g_2(x),$$

where g_1, g_2 and $\bar{f}_1$ are as above and $\bar{f}_2$ is bounded measurable.

Assume that $\bar{f}_1$ is antimonotone: for all $t \in [0,T]$ and $x, y \in \Omega$,

$$(\bar{f}_1(t,x) - \bar{f}_1(t,y))(x-y) \leq 0.$$

Let m and m' be two equilibria. Then, for almost all $t \in [0,T]$,

$$\int_\Omega g_1(x) m_t(dx) = \int_\Omega g_1(x) m'_t(dx).$$

The value of the representative agent is the same for all equilibria.

Outline

The limiting game

The reservoir size scales with the number of agents: $\widetilde{Z}_t = NZ_t$

$\Rightarrow$ each agent has a share Z_t which does not depend on N.

As $N \to \infty$, m_t^N converges to a deterministic limiting distribution m_t.

The proportion ω_t^N of the total demand which may be satisfied given the reservoir level converges to a deterministic proportion ω_t:

$$\omega_t = \frac{Z_t}{\int x m_t(dx)} \wedge 1.$$

The problem of individual agent becomes

$$\max_{\tau \in \mathcal{T}([0,T])} \mathbb{E}\Big[\int_0^\tau e^{-\rho t} p \omega_t M_t^{M_0} dt - \int_0^\tau e^{-\rho t} \hat{p}(1-\omega_t) M_t^{M_0} dt - e^{-\rho\tau} K + \int_\tau^\infty e^{-\rho t} \tilde{p} M_t^{M_0} dt \Big].$$

The limiting game

In the limit, our game becomes an optimal stopping MFG with reward functions

$$\tilde{f}(t,x,m) = x\left[(p+\hat{p})\left(\frac{Z_t}{\int_\Omega xm(x)dx}\wedge 1\right) - \hat{p}\right].$$

$$g(t,x) = \left\{-K + \frac{\tilde{p}x}{\rho-\mu}\right\},$$

so that

$$f(t,x,m) = xe^{-\rho t}\left[(p+\hat{p})\left(\frac{Z_t}{\int_\Omega xm(x)dx}\wedge 1\right) - \hat{p} - \frac{\tilde{p}\rho}{\rho-\mu}\right] + \rho K e^{-\rho t}.$$

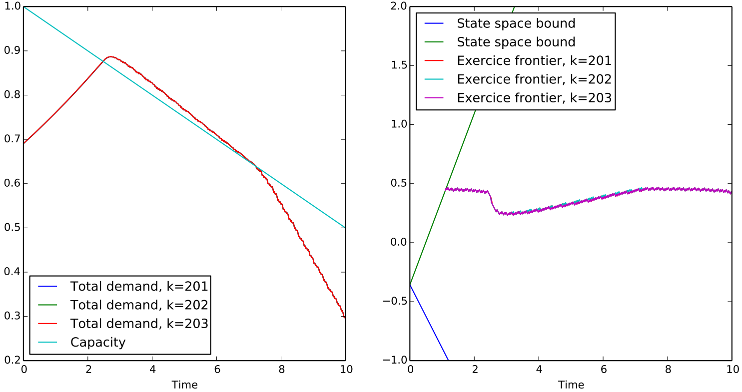
This problem satisfies the assumptions for existence and uniqueness

Since $f(t, X_t, m_t) \neq 0$ almost surely $\Rightarrow$ equilibrium with pure strategies

Numerical illustration

Production gain before switching	$p = 1$
Production gain after switching	$\tilde{p} = 1.4$
Penalty for not meeting the demand	$\hat{p} = 2.0$
Fixed cost of switching	$K = 3$
Discount factor	$\rho = 0.2$
Demand growth rate	$\mu = 0.1$
Demand volatility	$\sigma = 0.1$
Initial demand level	$M_0 = 0.7$
Reservoir capacity	$Z_t = 1 - 0.05t$
Time (latest possible switching date)	$T = 10$
Number of discretization steps	$N = 400$

Numerical illustration



Left: total demand and reservoir capacity as function of time. Right: Exercise frontier. To illustrate convergence, we plot three iterations of the algorithm.

Conclusion

Thank you for your attention!